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INTEGRAL REPRESENTATION OF POTENTIALS IN MEDIA WITH ADJOINING ANNULAR CRACKS AND SCREENS†

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For planar potential fields generalized matching conditions are derived corresponding to an arbitrary number of adjoining annular cracks and screens modelled by infinitely thin layers, the permeability of which is infinitely large for cracks and infinitely small for screens. The system of cracks and screens separates two piecewise-uniform zones. By applying the Fourier method followed by taking the convolution of Fourier integrals, integral representations of the potentials are constructed that satisfy generalized matching conditions in terms of arbitrary harmonic functions having singular points in the outer zone. The resulting formulae generalize the known results for a single crack or screen [1–3].

In many problems in the theory of filtration, heat conduction, electrostatics, etc. one must study potential fields in the vicinity of thin layers of large or small permeability [1–7], which can coexist in the same system. For example, when screening polluted zones from external flows it is advisable to supplement the screen by a drainage gap to direct the flow around the polluted zone and reduce both the pressure and the flow inside the zone. Similarly, it proves economical to place vertical drainage gaps and screens in a single system under a dam [5]. In addition, as a rule ideal contact cannot be maintained between different media, so that in a refined model one must take into account a transition layer consisting of open and sealed microlayers [2, 4].

Let $\varphi(r, \alpha)$ be a function satisfying the equation

$$r\partial_r(Kr\partial_r\varphi)+K\partial_\alpha^2\varphi=0,\ \ r\neq a \tag{1}$$

in a neighbourhood of the circle $r=a$, where $K=K_1+(K_2-K_1)s(r-a)$, $K_i>0$ are constants, r and α are the polar coordinates, $\partial_r^m=\partial^m/\partial r^m$, and $s(r)$ is the Heaviside unit function. By induction we shall derive generalized matching conditions for φ on the line $r=a$, which correspond to an arbitrary combination of cracks and screens adjoining one another modelled by the singularities of the coefficient K in (1) of the type $\delta(r-a)+1$ for cracks and $[\delta(r-a)+1]^{-1}$ for screens, $\delta(r)$ being the Dirac function. Denoting by φ^- and φ^+ the values of φ in the zones $D^-(r<a)$ and $D^+(r>a)$, respectively (on the line $r=a$, $\varphi^\pm$ being the limiting values), we consider generalized matching conditions of the form

$$r=a\colon\ [\varphi]=F_i(\varphi^-),\ [\upsilon_r]=G_i(\varphi^-) \tag{2}$$

where $[\cdot]$ is the jump of $\upsilon_r=K\partial_r\varphi$, and F_i and G_i are operators to be determined. When $F_i=G_i=0$ we have the classical matching conditions (ideal contact between two media).

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Suppose that conditions (2) and the following classical matching conditions are satisfied for $r=a$ and $r=b=a+l,\ l>0$

$$r=a:\ \varphi^0-\varphi^- = F_i(\varphi^-),\ \ \upsilon_r^0-\upsilon_r^- = G_i(\varphi^-)$$
$$r=b:\ \varphi^+ = \varphi^0,\ \ \upsilon_r^+ = \upsilon_r^0 \qquad (3)$$

Hence

$$\varphi^+|_b-\varphi^-|_a = \varphi^0|_b-(\varphi^0-F_i)|_a = \frac{l}{K_0}\upsilon_r^0\Big|_{c_1}+F_i\Big|_a \qquad (4)$$

$$\upsilon_r^0|_b-\upsilon_r^-|_a = \upsilon_r^0|_b-(\upsilon_r^0-G_i)|_a = K_0 l\partial_r^2\varphi^0|_{c_2}+G_i|_a \qquad (5)$$

where $c_j\in[a,\ b]$, and $K_0=\text{const}$, φ^0 and ϑ_r^0 are the values of K, φ and υ_r inside the layer $D_0(a<r<b)$, $\varphi|_a\equiv\varphi|_{r=a}$.

We take the limits as $l\to 0$ and $K_0\to\infty$ (the case of a crack $r=a+0$) in (4) and (5). It follows from (4), taking (3) into account, that $\varphi^0|_b\to\varphi^0|_a=(\varphi^-+F_i)|_a$. Thus for any $c\in[a,\ b]$ we have $\partial_r\varphi^0|_c\to 0$, or, taking (1) into account, $\partial_r^2\varphi^0|_c\to\partial_\alpha^2(\varphi^-+F_i)/a^2|_a$. Conditions (4) and (5) take the form (2), where

$$F_{i+1}=F_i,\ \ G_{i+1}=G_i-A_i a^{-2}\partial_\alpha^2(\varphi^-+F_i) \qquad (6)$$

$A_i=\lim lk_0$ being a parameter of the crack [7]. By analogy, when $l\to 0$ and $K_0\to 0$ (the case of a screen $r=a+0$), (5) and (3) imply that $\upsilon_r^0|_c\to\upsilon_r^-+G_i|_a$ for any $c\in[a,\ b]$, conditions (4) and (5) taking the form (2) where

$$F_{i+1}=F_i+B_i(G_i+K_1\partial_r\varphi^-),\ \ G_{i+1}=G_i \qquad (7)$$

and $B_i=\lim l/k_0$ is a parameter of the screen [7].

Therefore the generalized matching conditions have the form (2), where F_i and G_i for $i=0,$ $1, 2, \ldots$ are constructed from the recurrent formulae (6) and (7), in which $F_0=G_0=0$. In particular, for $i=1$, (2) implies the known conditions for a single crack and screen [1–3] obtained by a different argument. Below we assume that the cracks and screens with parameters A_{2k-1} and B_{2k} alternate for $k=1, \ldots, i$, $A_1\geqslant 0$, $B_{2i}\geqslant 0$, A_{2j+1}, $B_{2j}>0$ with $j=1, \ldots, i-1$, i being fixed.

Let the field be generated by those singular points of an arbitrary harmonic function $f(r,\alpha)$ for which $r>a$. Hence we have the problem (1), (2) for φ_i with $\varphi_i\sim f$. We represent φ_i by the Fourier series

$$\varphi_{ij}^- = \frac{f_0}{2}+\sum_{n=1}^{\infty}\sum_{\nu=1}^{2}a_{ij}^\nu\sigma_\nu\left(\frac{r}{a}\right)^n$$
$$\varphi_{ij}^+ = f(r,\alpha)+\sum_{n=1}^{\infty}\sum_{\nu=1}^{2}b_{ij}^\nu\sigma_\nu\left(\frac{a}{r}\right)^n;\ \ \sigma_1=\cos n\alpha,\ \sigma_2=\sin n\alpha \qquad (8)$$

where ν is an index, and $j=0$ or $j=1$, respectively, for $B_{2i}>0$ and $B_{2i}=0$. Thus, taking the Fourier expansions of f and $\partial_r f$ for $r=a$, from conditions (2) for $a_{ij}^\nu(n)$ and $b_{ij}^\nu(n)$ we obtain a system of equations, the solution of which has the form

$$a_{ij}^\nu = \frac{2K_2 a}{R_{ij}}f_\nu,\ \ b_{ij}^\nu = (-1)^j\left(1-\frac{2N_{ij}}{R_{ij}}\right)f_\nu$$
$$N_{i0}=(K_1+Q_i)a,\ \ N_{i1}=K_2(a+P_{i-1}n)$$
$$R_{ij}=a(K_1+K_2+Q_i)+K_2nP_{i-j} \qquad (9)$$

where f_ν are the Fourier coefficients of $f(a, \alpha)$ with respect to σ_ν, the parameters Q_i and P_i being constructed from the recurrent formulae

$$Q_q = Q_{q-1} + A_{2q-1}\frac{n}{a}\left(1 + P_{q-1}\frac{n}{a}\right) > 0$$
$$P_q = P_{q-1} + B_{2q}(K_1 + Q_q) > 0; \quad q = 1,\ldots,i; \quad P_0 = Q_0 = 0 \tag{10}$$

It follows that R_{ij} is a polynomial of degree μ in n, where $\mu = 2i - j$ for $A_1 > 0$ and $\mu = 2i - j - 1$ for $A_1 = 0$, the fractions in (9) being proper by virtue of (10) and the inequality $R_{ij} > 0$. (Here we take into account the relationships between the Fourier coefficients of f and $\partial_r f$ for $r = a$ that follow from the fact that $f(r, \alpha)$ is identical with the solutions of the Dirichlet and Neumann problems inside the circle $r < a$ with boundary functions $f(a, \alpha)$ and $\partial_r f(a, \alpha)$.) Since $f(a, \alpha) \in C^\infty$, it follows that $\varphi_{ij}^\pm \in Ch2(D^\pm)$ for all but the singular points, i.e. the functions given by (8) are solutions of the problem in question.

We can reduce (8) to quadratures. Let $R_{ij}(n)$ have simple real roots, i.e.

$$R_{ij} = S_{ij}\prod_{q=1}^{\mu}(n + \gamma_q)$$

where $\gamma_q > 0$, since $R_{ij}(n) > 0$. Representing $f(r, \alpha)$ for $r < a$ as the solution of the Dirichlet problem with boundary function $f(a, \alpha)$ by the Fourier method, multiplying the resulting expansion by $(r/a)^{\gamma-1}$ and integrating with respect to r, we obtain the formula

$$\sum_{n=1}^{\infty}\sum_{\nu=1}^{2}\frac{f_\nu\sigma_\nu}{n+\gamma}\left(\frac{r}{a}\right)^n = \Phi(r,\alpha,\gamma) - \frac{f_0}{2\gamma}, \quad r < a, \ \gamma > 0$$

where

$$\Phi(r,\alpha,\gamma) = \int_0^r\left(\frac{\xi}{r}\right)^\gamma \frac{1}{\xi}f(\xi,\alpha)d\xi \tag{11}$$

and an analogous formula with r replaced by a^2/r for $r > a$ in (11).

Decomposing (9) into simpler fractions, we write the potentials (8), apart from an additive constant, in the form

$$\varphi_{ij}^- = \frac{2K_2 a}{S_{ij}}\sum_{p=1}^{\mu}\frac{1}{M_p}\Phi(r,\alpha,\gamma_p) \tag{12}$$
$$\varphi_{ij}^+ = f(r,\alpha) + (-1)^j\left[f\left(\frac{a^2}{r},\alpha\right) - \frac{2}{S_{ij}}\sum_{p=1}^{\mu}\frac{N_{ij}(-\gamma_p)}{M_p}\Phi\left(\frac{a^2}{r},\alpha,\gamma_p\right)\right]$$
$$M_p = \prod_{q=1}^{\mu}(\gamma_q - \gamma_p)\Big|_{q\neq p}$$

(Φ has the form (11)).

If $R_{ij}(n)$ has a root $-\gamma_q$ of multiplicity n_q, then the potentials are determined by (12), in which one must take the limit as $\gamma_{q+p} \to \gamma_q$, $p = 1, \ldots, n_q$. In the case of complex roots $-\gamma_a$ the functions in (12) are real-valued. From the mathematical point of view formulae (12) provide integral representations of p-harmonic functions with a piecewise constant characteristic p having a combination of singularities of the type $(\delta + 1)^{\pm 1}$ on the circle $r = a$ in terms of harmonic functions f, retaining the type of singular points of the function f.

In the specific case of a crack $r = a - 0$ and a screen $r = a + 0$ with parameters A and B the potentials (12) for $i = 1$, $j = 0$ have the form

$$\varphi^- = \frac{2aK_2}{S}\int_0^r (t_1 - t_2)\frac{1}{\xi} f(\xi,\alpha)d\xi$$

$$\varphi^+ = f(r,\alpha) + f\left(\frac{a^2}{r},\alpha\right) - \frac{2}{S}\int_0^{a^2/r} (T_1 - T_2)\frac{1}{\xi} f(\xi,\alpha)d\xi$$

$$S = \rho(\gamma_2 - \gamma_1)/a, \quad \rho = K_2AB, \quad t_i = (\xi/r)^{\gamma_i}$$

$$\gamma_i = a\rho^{-1}[M + 2A + (-1)^i(M^2 - 4K_2\rho)^{1/2}]$$

$$T_i = (K_1a - A\gamma_i)(\xi r/a^2)^{\gamma_i}, \quad M = K_1K_2B - A$$

In the case when the singularities of the function f are distributed in the domain $r < a$ the solution of problem (1), (2) can be constructed in a similar way.

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